2 Distributions

Exercise 2.1. Show that the Dirac mass $\delta_0 : C_c(\mathbb{R}^d) \to \mathbb{R}$ defined as

$$\delta_0(\varphi) = \varphi(0)$$

is a continuous and linear functional on $C_c(\mathbb{R}^d)$ which can be extended, thanks to Hahn-Banach theorem, to a continuous and linear functional on $L^\infty(\mathbb{R}^d)$. Moreover, show that there exists no function $f \in L^1(\mathbb{R}^d)$ such that

$$\delta_0(g) = \int_{\mathbb{R}^d} f g \, dx \quad \text{for all } g \in L^\infty(\mathbb{R}^d).$$

Remark. From this exercise we deduce $\delta_0 \in L^\infty(\mathbb{R}^d)' \supsetneq L^1(\mathbb{R}^d)$.

Exercise 2.2. Show that the dipole functional $\delta_0' : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ defined as

$$\delta_0'(\varphi) = -\varphi'(0)$$

is a distribution, i.e. an element of $\mathcal{D}'(\mathbb{R})$. Moreover, show that there exists no measure μ such that

$$\delta_0'(\varphi) = \int_{\mathbb{R}} \varphi \, d\mu$$

Remark. From this exercise we deduce that the set of measures is strictly contained in $\mathcal{D}'(\mathbb{R})$.

Exercise 2.3. Let $\Omega \subset \mathbb{R}^d$ be an open set and $\{T_n\}_{n\in\mathbb{N}} \subset \mathcal{D}'(\Omega)$ be a sequence of distributions which is weakly converging to $T \in \mathcal{D}'(\Omega)$. Show that for any multi-index α the sequence $\{D^\alpha T_n\}_{n\in\mathbb{N}}$ is weakly converging to $D^\alpha T$.

Exercise 2.4. For every $n \in \mathbb{N}^*$ consider the distributions in $\mathcal{D}'(\mathbb{R})$ given by

$$T_n = \delta_{\frac{1}{n}} \quad \text{and} \quad S_n = n(T_n - T_{2n}).$$

Compute the respective weak limits of $\{T_n\}_{n\in\mathbb{N}^*}$ and $\{S_n\}_{n\in\mathbb{N}^*}$ in $\mathcal{D}'(\mathbb{R})$.

Exercise 2.5.

1. Compute the distribution $\exp \delta_0''$.
2. Given $a, b > 0$, compute the distributional derivative of

$$f_{a,b}(x) = H(x) \log|a\,x| + H(-x) \log|b\,x|,$$

where $H(x)$ indicates the Heaviside function.